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A Bayesian copula model for stochastic claims reserving

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Abstract

We present a full Bayesian model for assessing the reserve requirement of multiline Non-Life insurance companies. Bayesian models for claims reserving allow to account for expert knowledge in the evaluation of Outstanding Loss Liabilities, allowing the use of additional information at a low cost. This paper combines a standard Bayesian approach for the estimation of marginal distribution for the single Lines of Business for a Non-Life insurance company and a Bayesian copula procedure for the estimation of aggregate reserves. The model we present allows to "mix" own-assessments of dependence between LoBs at a company level and market-wide estimates provided by regulators. We illustrate results for the single lines of business and we compare standard copula aggregation for different copula choices and the Bayesian copula approach.

Keywords : stochastic claims reserving; bayesian copulas; solvency capital requirement; loss reserving; bayesian methods
JEL classification : C11, G22

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1 Introduction

The estimation of Outstanding Loss Liabilities (OLLs) is crucial to reserve risk evaluation in risk management. Classical methods based on run-off triangles need a small amount of input data to be used. This fact determined their fortune, making them immediate to use, requiring the knowledge of triangle of annual paid claims amount only. However, this fact constitutes also an important shortcoming, since using a small sample of data to predict future outcomes may possibly lead to inaccurate estimates. Anyway, their widespread use in professional practice encourages further improvements to limit this problem.

Starting from the beginning of this century, bayesian methods in estimating run-off triangles gained increasing attention as a tool to include expert judgement in stochastic models[1] and enlarge the information set on which reserves are computed. The use of Bayesian methods in loss reserving started decades ago, but it was the possibility of using MCMC fast computer-running algorithms that gave high flexibility to the application of this methodology, allowing for almost unrestricted distributional assumptions. De Alba (2002), De Alba and Nieto-Barajas (2008) - who introduced correlation among different accident years - and Ntzoufras and Dellaportas (2002) offer examples of how Bayesian methods can be implemented in the estimation of outstanding claims for a line of business, introducing prior information on both future claim amount (ultimate costs) and frequency. Simultaneously, some works tried to introduce the use of copulas - which gained increasing popularity in the finance world in the last decade - also in loss reserving[2].

The question of how to cope with dependent risks such as the losses an insurance company has to face in its different lines of business (LoBs) is surely of utmost importance. Current practice and Solvency II standard formulas account for diversification by means of linear correlation matrices estimated on a market-wide basis. Obviously, these correlation matrices can fail to capture the specificities insurance companies can present, due to geographical reasons or management choices.

A few papers studied the application of copulas to run-off triangles estimation. Tang and Valdez (2005) used simulated loss ratios to aggregate losses from different LoBs. Li (2006) compared aggregation through the use of different copula functions, given distributional assumptions on the marginals. More recently, De Jong (2009) introduced a Gaussian copula model to describe dependence between LoBs.

[1]For a nice treatment on the use copulas to aggregate expert opinions, see for example the seminal work Jouini and Clemen (1996).

[2]Copulas have been recently used in individual claim models (Zhao and Zhou (2009)).

This paper aims at combining both these two important aspects: bayesian methods and the use of copulas. The bayesian approach, introducing data coming from expert judgement, allows to include additional reliable information when estimating reserves and to derive full predictive distibutions. Copulas allow to obtain joint distributions in an easily tractable way, separating the process of defining the marginals and the dependence structure. Hence, we introduce prior information on the dependence structure, using Bayesian copulas in the aggregation of losses across LoBs. Up to our knowledge, this paper is the first attempt in introducing Bayesian copulas in stochastic claims reserving. Dalla Valle (2009) applied a similar technique to the problem of the estimation of operational risks. We adapt their approach to the aggregation of OLLs from different LoBs.

Combining a Bayesian approach to derive the marginal distributions of OLLs for each single LoB and the use of Bayesian copulas to aggregate them, one obtains a fully Bayesian model that incorporates expert judgement on the ultimate costs and development pattern of each LoB as well as on the dependence structure between them.

We apply this model to four lines of business of an Italian insurance company. We compare results obtained from the Bayesian copula model with those obtained from a standard copula approach.[3].

The outline of the paper is the following. Section 2 presents a simple Bayesian model, which uses Markov Chain Monte Carlo (MCMC) simulation methods to derive the predictive distribution of OLLs for each LoB. Section 3 motivates the choice of modeling dependence between LoBs and briefly reviews the most important notions on the theory of copulas. Section 4 presents the Bayesian copula approach. Section 5 applies the model to a large insurance company, reports and compares the results. Section 6 concludes.

2 A Bayesian approach for computing Lob's reserves

First, we briefly present a bayesian model for the estimation of the OLLs for single LoBs. We assume that an Over-Dispersed Poisson (ODP) distribution models incremental claims in the run-off triangle. Then, denoting with X_{ij} the claim payments in the development year (d.y.) j concerning accident

[3]Financial literature offered only few examples of application of non bivariate copulas. This paper, testing the theoretical framework on a multi-line insurance company, provides a four-dimensional application of our model of aggregation through copulas and a comparison of results for different copula choices

year (a.y.) i and with ϕ_i the overdispersion coefficient for accident year i, we assume that $\frac{X_{ij}}{\phi_i}$ are independently Poisson distributed with mean $\frac{\mu_i\gamma_j}{\phi_i}$:

$$\mathbb{E}\left[\frac{X_{ij}}{\phi_i}|\Theta\right] = \frac{\mu_i\gamma_j}{\phi_i},$$
$$\mathrm{Var}\left[\frac{X_{ij}}{\phi_i}|\Theta\right] = \frac{\mu_i\gamma_j}{\phi_i},$$
$$\phi_i > 0, \mu_i > 0 \;\forall\; i = 1, ..., I, \gamma_j > 0 \;\forall\; j = 1, ..., J,$$

$$\Theta = \mu_1, ..., \mu_I, \gamma_1, ..., \gamma_J, \phi_1, ..., \phi_I,$$

We renormalize the model setting the (observed) parameter $\mu_1 = 1$. μ_i' s then represent ultimate claims relative to year 1, while the γ_j' s represent the development pattern in monetary terms relative to ultimate cost of a.y. 1. This renormalization allows to increase flexibility in distributional assumptions, avoiding the awkward constraint that the parameters of the development pattern by d.y. have to sum up to 1.
We estimate the overdispersion parameter ϕ using the Pearson's residuals obtained from the triangle and assume it constant across accident years. Although one can object that this prevents the model from being fully bayesian, this choice is backed by two important considerations: first, ϕ has hardly a simple economic interpretation and, consequently, it will be hard to define a reasonable prior distribution to model it. Moreover, the MCMC algorithm turns out to be considerably more stable if ϕ is not bayesian. We choose the prior distribution of both the μ's and the γ's to be independently gamma distributed. We don't have analytical expressions for the posterior distribution. Hence, we set up a Markov Chain Monte Carlo algorithm in order to simulate the posterior distribution of parameters. The prior distribution is set through coefficient of variation (cv) and mean; at each step, we update the parameters to match the current mean values of $\mu^{(t)}$ and $\gamma^{(t)}$, where t is the iteration step in the simulation algorithm. Hence,

$$\mu_1 = 1$$
$$\mu_i \sim \Gamma(a, b_i) \; i = 2, ..., I$$
$$\text{with } a = \frac{1}{cv(\mu)^2} \text{ and } b_i = \frac{\mu_i^P}{a}$$

and

$$\gamma_j \sim \Gamma(c, d_j) \; j = 1, ..., J$$

$$\text{with } c = \frac{1}{cv(\gamma)^2} \text{ and } d_j = \frac{\gamma_j^P}{c}$$

We implement the MH algorithm with gamma proposal distributions, whose coefficient of variation is kept fix throughout the algorithm. The lower part of the triangle is obtained through simulation and then discounted using the term structure of interest rates at the end of the last a.y..

3 A copula approach to aggregate across LoBs

In the previous section we presented a way of retrieving the predictive distribution of OLLs for a single line of business. From now on, we address the problem of generating the joint distribution of OLLs from different LoBs, in order to estimate prudential reserves for multi-line insurance companies. Notice that what follows can be applied independently of the choice of the method used to obtain the predictive distribution of reserves.
Correctly capturing the presence of dependence between the losses in different LoBs is intuitively a desirable feature of a good model for claims reserving. The following Table compares the correlation matrix between the LoBs of an Italian insurance company, estimated from a time series of loss ratios, and the one the CEIOPS mandated to use when calculating reserve risk with the standard formula in the Quantitative Impact Studies (QIS):

	LoB			
	MTPL	**MOC**	**FP**	**TPL**
MTPL	1 (0)	0.4751 (0.0463)	0.4598 (0.0549)	0.5168 (0.0281)
MOC	0.4751 (0.0463)	1 (0)	0.8789 (0.000001)	0.7331 (0.0005)
FP	0.4598 (0.0549)	0.8789 (0.000001)	1 (0)	0.8748 (0.000002)
TPL	0.5168 (0.0281)	0.7331 (0.0005)	0.8747 (0.000002)	1 (0)

Table 1: Linear correlation between LoBs. The brackets report p-values.

Table 1 clearly shows that the "industry-wide" estimate proposed by CEIOPS and the industry-specific ones differ. Results on the correlation of a time series of realized losses, which we will present in Section 5 further support this evidence.
We then turn to the use of copulas in order to model the dependence between

LoB				
	MTPL	**MOC**	**FP**	**TPL**
MTPL	1	0.25	0.25	0.5
MOC	0.25	1	0.5	0.25
FP	0.25	0.5	1	0.25
TPL	0.5	0.25	0.25	1

Table 2: This Table reports the correlation matrix the CEIOPS estimated and requires the participants to the Quantitative Impact Studies (QIS) to use in the evalutaion of reserves.

LoBs. Indeed, they allow us to separate the estimation of the characteristics of the dependence structure from the modeling of marginal distributions.

3.1 Copulas

In this section we briefly give the basic definitions and fundamental notions about copulas. We are interested in modelling the joint distribution $F(L_1, ..., L_n)$, where L_i denotes the OLLs of the i-th LoB of a company whose business involves n sectors, since our object of interest is

$$L_{tot} = \sum_{i=1}^{n} L_i$$

and its related percentiles. Copula functions permit us - as we will briefly show in this section - to separate the process of estimating the marginal distributions $F(L_1), ..., F(L_n)$ of the OLLs of each LoB from the estimation of the dependence structure. Moreover, the latter can be modeled in a highly flexible way, since many copula functions are available to describe it and capture its (also non-linear) properties. We recall the most important results on multivariate copulas, which we will use in the construction of our model.[4] First of all, we define multivariate copulas:

Definition 3.1 *An n-dimensional subcopula is a function* C:$A_1 \times A_2 \times ... \times A_n \to \mathbb{R}$*where, for each* $i, A_i \subset I$ *and contains at least 0 and 1, such that:*

1. *C is grounded*[5]

2. *its one-dimensional margins are the identity function on I: $C_i(u) = u$, $i = 1, 2, ..., n$*

[4]For a comprehensive review of the theory of copulas and their use in finance the reader can refer to Cherubini, Luciano, and Vecchiato (2004) and Nelsen (2006).

[5]Let $C : \mathbb{R}^{*n} \to \mathbb{R}$ be a function with domain $A_1 \times ... \times A_n$, where A_i are non-empty sets with a least element a_i. C is grounded iff it is null for every $\mathbf{v} \in \text{Dom } C$, with at least

3. *C is n-increasing.*[6].

C is a copula if it is an n-dimensional subcopula for which $A_i = I$ for every i.

The following (Sklar's) theorem proofs the link between a copula and the marginal distribution functions[7]:

Theorem 3.2 *Let $F_1(x_1), ..., F_n(x_n)$ be marginal distribution functions. Then, for every $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^{*n}$:*

1. *If C is any subcopula whose domain contains $RanF_1 \times ... \times RanF_n$, $C(F_1(x_1), ..., F_n(x_n))$ is a joint distribution function with margins $F_1(x_1)$, ..., $F_n(x_n)$.*

2. *Conversely, if F is a joint distribution function with margins $F_1(x_1)$, ... ,$F_n(x_n)$ there exists a unique subcopula C, with domain $RanF_1$ $\times ... \times RanF_n$ such that $F(\mathbf{x}) = C(F_1(x_1), ..., F_n(x_n))$.*

If the marginals are continuous, the subcopula is a copula; if not, there exists a copula $\mathcal{C}$ such that

$$\mathcal{C}(u_1, ..., u_n) = C(u_1, ..., u_n) \text{ for every } (u_1, ..., u_n) \in RanF_1 \times ... \times RanF_n.$$

This is the most important result in the theory of copulas: it states that - as we pointed out before - starting from separately determined marginals and dependence structure copula functions allow to represent a joint distribution function of the variables involved. Moreover, overall uniqueness is ensured when marginals are continuous; when they are discrete, uniqueness is guaranteed on the domain $RanF_1 \times ... \times RanF_n$.

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one index k such that $v_k = a_k$:

$$C(\mathbf{v}) = C(v_1, v_2, ..., v_{k-1}, a_k, v_{k+1}, ..., v_n) = 0$$

[6] $C : A_1 \times A_2 \times ... \times A_n \to \mathbb{R}$ is n-increasing if:

$$\sum_{\mathbf{w} \in ver(A)} C(\mathbf{w}) \prod_{i=1}^{n} sgn(2w_i - u_{i1} - u_{i2}) \geqslant 0$$

where *ver(A)* is the set of vertices of A.

[7] For a proof of this theorem in the multivariate case we refer the reader to Schweizer and Sklar (1983).

3.2 Applying copulas to claims reserving

We outline first a simple procedure to obtain a joint distribution of OLLs for an n-dimensional non-life insurance company through the use of copulas:

1. derive the marginal distributions of the OLLs $F(L_1), ..., F(L_n)$ for each LoB independently. For this task, it is possible to resort to classical methods, simulation, as well as to the Bayesian technique we described in Section 2.[8]

2. estimate the dependence structure between the L_i's for $i = 1, ...n$.

3. choose a convenient copula function and evaluate its parameter(s). The copula will satisfy the uniqueness properties of Theorem 3.2, depending on the form of its marginals.

Sampling from any n-dimensional copula obtained can be done exploiting the properties of conditional distributions. Then, we can easily evaluate the quantities of interest on the simulated sample. Difficulties in the procedure above lie mainly on the correct estimation of the dependence structure, which is a complicated task given the low (annual) frequency of the input data used in stochastic claims reserving models based on run-off triangles. The same observation applies to the choice of the most appropriate copula function. In section 5.2 we compare the results of evaluating the OLLs of a multi-line insurer under different copula assumptions.

4 A Bayesian copula approach

As we saw in Tables 1 and 2, industry-wide estimates of the dependence between the different LoBs and own assessments based on companies' tracks can differ. This can be due - for example - to geographical issues as well as to management actions or policies. Hence, including company-specific measures of dependence in reserves' estimation as expert judgement together with industry-wide estimates can help in improving the quality of the predictions of future losses. Hence, we present a Bayesian approach to the use of copulas, by adding uncertainty on the parameters of the copula function.
The procedure which has to be applied to implement the Bayesian copula model is similar to the one we described for standard copulas in the previous Section 3.2, but a few more steps are required:

1. choose a convenient distributional assumption for the prior of the copula parameter(s) θ, $\pi(\theta)$

[8] Tang and Valdez (2005) used simple distributional assumptions on the marginals

2. compute, using Bayes' theorem, the posterior distribution of the parameter given the input data:

$$f(\theta|\mathbf{x}) = f(\mathbf{x}|\theta)\pi(\theta),$$

where $\mathbf{x}$ denotes the $n \times T$ matrix of observations (T is the number of observations).

A convenient choice of the prior distribution requires the choice of priors whose densities are conjugate to the one of the distribution of the estimation object - in our context, OLLs per a.y.. We now present - as an example - the application of the procedure to a Gaussian copula choice.

4.1 Bayesian Gaussian copula

This Section introduces the use of Bayesian Gaussian copulas for aggregating marginal distribution of OLLs. The choice of a Gaussian copula is the most immediate one and it entails using a linear measure of dependence - linear correlation - to represent the link between LoBs. Hence, we assume that OLLs across different LoBs are distributed following a multivariate Gaussian density. The multivariate Gaussian copula density is

$$c(u_1, ..., u_n|\Sigma) = |\Sigma|^{-1/2} exp\left\{-\frac{1}{2}\mathbf{x}'(\Sigma^{-1} - I_n)\mathbf{x}\right\}$$

where Σ denotes the $(n \times n)$ covariance matrix between the LoBs, I_n is an identity matrix of dimension n and $\mathbf{x}$ is a matrix of observed OLLs. Dalla Valle (2009) applied Bayesian Gaussian copulas to the estimation of operational losses. However the paper considered correlation matrices, incurring in the problem of requiring the priors to be vague - and thus uninformative. Starting from equation 4.1, we take the product over the T observation of the sample to obtain the likelihood function:

$$f(\mathbf{x}|\Sigma) = |\Sigma|^{-T/2} exp\left\{-\frac{1}{2}\sum_{i=1}^{T}(\mathbf{x_i} - \mathbf{m})'(\Sigma^{-1})(\mathbf{x_i} - \mathbf{m})\right\}$$

We choose the Inverse Wishart as a prior distribution for Σ. Inverse Wishart distributions are commonly used to represent covariance matrices and the attractive property of being conjugate to the multivariate Gaussian. Hence we set

$$\pi(\Sigma) \sim W^{-1}(\alpha, B).$$

, where $W^{-1}(\alpha, B)$ denotes the Inverse Wishart Distribution with α degrees of freedom and precision B. We can write its probability density function

$$f(\Sigma|\alpha, B) = \frac{|B|^{\alpha/2} \, |\Sigma|^{-(\alpha+n+1)/2} \, e^{-tr(B\Sigma^{-1})}}{2^{\alpha n/2}\Gamma_n(\alpha/2)}$$

and apply Bayes' theorem to compute its posterior distribution.

A standard result in bayesian statistic (see e.g. DeGroot (2004)) allows us to conclude that the posterior distribution of Σ follows again an inverse Wishart distribution:

$$\pi(\Sigma|\mathbf{x}) \sim W^{-1}(T + \alpha, B + \sum_{i=1}^{T} \mathbf{x}_i \mathbf{x}'_i)$$

Hence, the precision parameter of the posterior inverse Wishart is then given by the sum of the precision parameter in the prior distribution and $T-1$ times the sample covariance matrix. Since the mean of the inverse Wishart distribution is

$$\mathbb{E}[\Sigma] = \frac{B}{\alpha - n - 1}$$

Once set the two parameters of the prior distribution and computed the sample covariance, it is possible to simulate a random sample from the Gaussian copula for OLLs, drawing first a covariance matrix from the inverse Wishart distribution and then use it to generate outcomes from the Gaussian copula. Then, we generate a sufficient number of outcomes from the Bayesian Gaussian copula and compute the distribution of aggregate OLLs.

5 An application to an Italian Insurance Company

In this section we apply the methodologies described in the previous parts of the paper to obtain and compare the predictive OLLs distribution of a multi-line insurance company for different (standard and bayesian) copula choices. First, in section 5.1 we derive the marginal distribution for each LoB as described in Section 2, then we compare the results obtained by aggregating the marginals using both standard and Bayesian copulas. Our dataset is composed by the paid claims triangles of a large insurance company from 2001 to 2008. We restricted our attention to its 4 most important LoBs, whose linear correlation estimates based on a time series of loss ratios - reported in Table 1 - were significant at least at a 10% level.

It is important to remark that we derive full predictive distributions and, as a consequence, that we can easily compute not only best estimates, but all the relevant percentiles[9]. Notice that the approach can be easily extended to derive a predictive distribution of the overall losses of a company, considering all the LoBs in which it is involved. However, we recognize that data quality must be high enough to return reliable estimates of the dependence structure.

5.1 Estimation of the marginals

Applying the method described in section 2 we derive the marginal distribution of OLLs 4 LoBs: Motor Third Party Liability (MTPL), Motor Other Classes (MOC), Fire and Property (FP) and Third Party Liability (TPL). Table 3 reports the most important figures of the predictive distribution for different choices of the precision of the priors. We define the mean of the prior distribution of ultimate costs (before renormalization) as

$$\mathbb{E}[\mu_i'] = \sum_{j=1}^{r-i+1} X_{ij} + R^S(i)$$

This means that, for each a.y., the prior mean is given by the sum of the last observed cumulative claim payment and the statutory reserves $R^S(i)$. The mean prior on the development pattern is simply the chain ladder estimate:

$$\mathbb{E}[\gamma_j'] = \beta_j^{CL} - \beta_{j-1}^{CL}$$

where

$$\beta_j^{CL} = \prod_{k=j}^{J-1} \frac{1}{\hat{\lambda}_k}, k = 1, ..., J-1$$

and $\hat{\lambda}_k$ are the chain ladder estimates of the development factors. Model parameters' prior mean μ_i, i=2,...,I and γ_j, j=1,...J are obtained as

$$\mathbb{E}[\mu_i] = \frac{\mathbb{E}[\mu_i']}{\mu_1}$$

$$\mathbb{E}[\gamma_j] = \mathbb{E}[\gamma_j'] * \mu_1$$

Table 3 compares the key figures of the predictive distribution for each LoB for different choices of the coefficient of variation of the priors.

It shows largely different OLL distributions for different prior choices. As soon as the priors become more vague the estimates converge to the

[9]This is important in terms of the VaR-approach followed by Solvency II.

chain ladder ones, while convergence to the prior is achieved when the priors themselves are precise. Hence, differences arise, since statutory reserves are not computed using the Chain Ladder method, but a different one which accounts for the speed of finalization and mean costs also. It is easy to notice from the Table that - as one could expect - as soon as the prior information becomes less precise the standard deviation of the distribution of OLLs increases.

5.2 Estimation of reserves through copulas

In this Section we present model results obtained from classical copula methods and compare the figures obtained with different copula choices. We first estimate copula parameters from adequately chosen time series data. We decided to use loss ratios, following Tang and Valdez (2005), since they constitute the most reliable source of information. Our choice of using the historical loss ratio series is motivated by the lack of qualitatively useful data about our direct object of interest, losses, for which historical data are either unavailable or too far away in time.[10] Hence we implicitly assume that the correlation between loss ratios is a good proxy for the correlation of losses themselves. Moreover, loss ratios are a non-monetary measure, allowing us to abstract from the challenges of correctly capturing overall and LoB-specific inflation when estimating.
While Tang and Valdez (2005) used industry-wide estimates, we use a company-specific time series of loss ratios. We compare the results obtained from these industry-specific estimates with those obtained using the matrix proposed by the CEIOPS in the Quantitative Impact Study 5 (QIS 5).

We first deal with the Gaussian and the t copulas, using the (ML) linear correlation estimated matrix reported in Table 1 as the parameter. The Upper Panel of Table 4 compares the results from the Independence, the Gaussian and the Student's t copula with 4 degrees of freedom.

[10]In Section 5.3, however, we will be forced to derive a measure of losses per a.y. using observed data and statutory reserves

Precise Priors (cv 0.05)				
LoB	**4**	**5**	**7**	**8**
Mean	404388773.3	13519713.9	34505262.	65461545
Std Deviation	9732942.751	441754.4416	1081978.667	1580984.48
VaR 75	410995838.9	13816484.7	35222255.7	66524262.8
VaR 97.5	423699508	14392490.3	36666395.9	68564682.2
VaR 99	427297420.1	14567031.5	37072985.5	69171034.6
VaR 99.5	429308413.2	14681856.6	37329907.2	69600078.3
RC	24919640.0	1162142.6	2824644.6	4138533.1
Intermediate Priors (cv 0.1)				
LoB	**4**	**5**	**7**	**8**
Mean	433049373.5	13748473.7	35500766.5	68943031.8
Std Deviation	14963435.3	515438.8	1436387.9	2699322.4
VaR 75	442997813.3	14099498.8	36452316.5	70747579.4
VaR 97.5	463012090.3	14776090.2	38371337.7	74270158.7
VaR 99	469270928.2	14972190.1	38966614.0	75294702.3
VaR 99.5	473098522.3	15109019.8	39376102.7	76096070.4
RC	40049148.8	1360546.2	3875336.2	7153038.6
Vague Priors (cv 0.5)				
LoB	**4**	**5**	**7**	**8**
Mean	466678568.2	13823953.1	36158247.4	76376715.6
Std Deviation	23091344.0	589537.2	1798075.8	4849745.317
VaR 75	482416095.4	14222419.8	37337075.1	79526592.3
VaR 97.5	513180633.3	15015262.8	39834101.3	86254875.9
VaR 99	521989486.5	15246807.0	40609193.4	88270812.9
VaR 99.5	528830674.6	15412128.7	41120258.5	89657400.2
RC	62152106.4	1588175.7	4962011.2	13280684.6
Very Vague Priors (cv 1.5)				
LoB	**4**	**5**	**7**	**8**
Mean	469457550.6	13889439.6	36537741.4	77348490.4
Std Deviation	23657712.1	631536.8	1930043.5	4823628.448
VaR 75	485141513.3	14297531.7	37813249.2	80594262.3
VaR 97.5	516938239.6	15181185.6	40468961.1	87318681.5
VaR 99	527327189.6	15420777.1	41177121.6	89221792.3
VaR 99.5	534835778.5	15568471.2	41727870.5	90309880.2
RC	65378227.9	1679031.6	5190129.0	12961389.8

Table 3: This Table reports mean and VaR measures for the OLL distributions for each LoB for different choices of cv of the priors

Figures	**Copula Type**				
	Independence	**Gaussian MLE**	**Gaussian QIS**	**Student t MLE**	**Student t QIS**
Mean	593012692.6	592999468	593078422.2	593107009.4	593089632.8
Std Dev	23661871.8	27330843.4	26588306.7	27336552.8	26466577.4
VaR 75	609074201.8	611478073.8	610981167.6	611481550.3	610664068.4
VaR 97.5	640745885.5	647956557.4	646600022.3	648294764.2	646824832.7
VaR 99	649693108.9	658180123.7	657255148.1	659325286.3	658173341.8
VaR 99.5	656323827.6	665951381.4	664323502.5	666734284.0	665631981.4

Figures	**Archimedean Copula Type**		
	Clayton	**Frank**	**Gumbel**
Mean	592311157.2	592329320.5	592142996.5
Std Dev	25889681.9	25571649.1	25744039.5
VaR 75	610002820.2	609871400.4	609129608.0
VaR 97.5	642228178.2	643249755.5	644850509.7
VaR 99	651706417.7	652817211.1	656423299.3
VaR 99.5	657929168.7	659126935.2	664429091.2

Table 4: This Table compares mean, standard deviation and various percentiles of the joint OLL distribution of the 4 LoBs, obtained from different copula choices. The Upper Panel reports results from the Independence, the Gaussian and the t copula. In the "MLE" columns the parameter of the copula is the Maximum Likelihood estimator of the correlation matrix - the sample correlation - while in the "QIS" column the QIS5 correlation matrix reported in Table 2 was used as the copula parameter. The Lower Panel compares figures from different Archimedean copula choices.

Since no negative correlation between the LoBs is captured by the time series of loss ratios nor by the QIS 5 matrix, the Independence copula obviously offers the lowest level of Prudential Reserves (656 millions). Using the own assessment of correlation (in the Gaussian or t MLE column) returns a distribution with slightly fatter tails than the one obtained when using the CEIOPS QIS matrix. The consequent "risk capital" - which is the capital kept in excess of the best estimates of OLLs - is also higher when using the own assessment of correlation. We then turn our attention to Archimedean copulas. First, we estimate the Kendall's τ matrix from the time series of loss ratios. Since the corresponding parameters of the bivariate copulas involving the various LoBs differ, we used the recursive procedure of Genest (1987) and Genest and Rivest (1993) in order to generate random samples from the multivariate copulas. The procedure exploits the properties of conditional distribution functions. We describe the technical details and the algorithm of the procedure in the Appendix. The Lower panel of Table 4 reports also the figures obtained when using the Archimedean Clayton, Gumbel and Frank's copulas.

The 99.5th percentile of the OLL distribution - which is usually indicated as a standard measure for prudential reserves in the Solvency II framework - computed with the Clayton copula is lower than the one obtained with any other copula type, with the only exception of the Independence copula. On the contrary, Gumbel's copula predicted $VaR_{99.5\%}$ is the highest among the Archimedean families we compared, but its estimate is however lower than the one obtained from the Gaussian and the t copulas.

5.3 Estimation of reserves using Bayesian Copulas

Unfortunately, the procedure described in Section 4 can not be applied when the record of past losses is not sufficiently long and homogenous across years. In our data sample, information about past claim payments lacked the deepness to allow us to use a significant time series of observed OLLs. To overcome this problem, we derived a time series of losses adding the observed paid claims by a.y. for the d.y. available (inflated at a monetary rate of inflation provided by ISTAT) and the reserved amount at the end of the observation period obtained from the balance sheet. Using this time series, we obtained an own assessment of correlation which is reported in Table 5. This matrix evidently differs sharply from both the QIS5 one and the one estimated from the time series of loss ratios. In particular, losses in the MTPL LoB appear to be negatively correlated with the other 3 LoBs, which, as happened in terms of loss ratios, show instead a very high degree of positive correlation. It is worth noticing however that these estimates, computed on a small series

of data, can be inaccurate, as the p-values in the Table highlight. Hence, the idea of coupling this information with some more reliable assessments, such as the market-wide one provided by some the CEIOPS seems particularly appropriate.
We assume that losses across LoBs follow a multivariate Gaussian distribution. We include uncertainty on the variance/covariance matrix and we assume a prior Inverse Wishart distribution with precision parameter equal to the QIS5 implied variance/covariance matrix[11] and $n+2$ degrees of freedom. As we showed in the previous section, we can then derive easily the posterior distribution of this variance/covariance matrix and sample from it to generate the multivariate Gaussian copula outcomes. This posterior distribution accounts for both the mean prior (the QIS5 matrix) and the estimated variance/covariance matrix of Table 5. We then obtained the aggregate distribution of OLLs, using the "vague priors" marginals derived in 2.
Table 6 reports best estimates and relevant quantiles of predicted OLLs.

	LoB			
	MTPL	**MOC**	**FP**	**TPL**
MTPL	1 (0)	-0.5275 (0.1791)	-0.5389 (0.1682)	-0.3530 (0.3910)
MOC	-0.5275 (0.1791)	1 (0)	0.9728 (0.000001)	0.8945 (0.0027)
FP	-0.5389 (0.1682)	0.9728 (0.000001)	1 (0)	0.8560 (0.0067)
TPL	-0.3530 (0.3910)	0.8945 (0.0027)	0.8560 (0.0067)	1 (0)

Table 5: Linear correlation between LoBs estimated from a time series of losses.Brackets report p-values.

We first compare standard Gaussian copula results when using this new estimated matrix with those obtained using the previously reported assessments of correlation. The distribution of joint OLLs obtained using a standard Gaussian copula with the matrix showed in 5 as a parameter is leptokurtic with respect to the one obtained assuming an Independence copula. The $VaR_{99.5\%}$, in particular, is 7 millions lower (649 vs. 656 millions of euros).
The Bayesian Gaussian copula approach, instead, "mixing" between the use of this own assessment of correlation and the QIS matrix reported in Table 2, presents a less sharp diversification effect (its $VaR_{99.5\%}$ is 652 millions of

[11]We get this precision matrix by transforming the correlation matrix using the estimated sample variance

Figures	Copula Type		
	Independence	**Gaussian MLE**	**Bayesian Gaussian**
Mean	593012692.6	593072251.84	593108913.85
Std Dev	23661871.8	21099212.30	21859435.62
VaR 75	609074201.8	607307094.19	607782835.70
VaR 97.5	640745885.5	635540055.89	637227064.46
VaR 99	649693108.9	643711062.03	646107459.86
VaR 99.5	656323827.6	649551647.17	652061589.16

Table 6: This Table reports the simulated joint distribution of the OLLs for an Independence copula, a Gaussian copula whose MLE of the correlation matrix is estimated from the time series of losses as described in Section 5.3 and for the Bayesian Gaussian copula.

euros). Figures 1 and 2 compare the densities of the predictive distribution of OLLs obtained from the standard Gaussian copula with the estimated and the QIS correlation matrix and the one obtained with the Bayesian Gaussian technique. They clearly show that this latter distribution is "intermediate" between the other two. Its standard deviation increases with respect to the one reported in the Gaussian MLE, as reported in Table 2. The distribution obtained using a Gaussian copula with the QIS 5 correlation matrix as the parameter presents, as expected, fatter tails with respect to the Bayesian Gaussian one. This is due to the different dependence structure and mainly to the negative correlation between the biggest LoB in terms of volume (the MTPL one) and the other ones.

6 Conclusions

In this paper we proposed a way to couple Bayesian methods and copulas for stochastic claims reserving method. We showed how to account for expert judgement through Bayesian techniques not only in the estimation of the marginal distribution of losses, but also in the process of aggregating these estimates across multiple Lines of Business.
We made use of copula functions, which allowed us to treat separately the marginals and the dependence structure. We examined how to introduce uncertainty on copula parameters. In particular - due to their analytical tractability - we focused on Bayesian Gaussian copulas. We presented an application of the methodology to a large multi-line Italian insurance company and we compared the results obtained with standard copula aggregation - under different assumptions on the copula type - and the Bayesian copula model.

Unfortunately, at the moment, lack of enough statistically reliable data mine the possibility of backtesting the model and the performance of goodness of fit tests for establishing best fit copulas. However, we plan to explore these aspects in the future. Extension of the Bayesian approach to other copula functions (t or Archimedean) can also be addressed.

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7 Appendix - Generating n-dimensional copulas

7.1 The Genest and Rivest approach

In this Appendix we present Genest and Rivest's approach to the simulation of n-dimensional Archimedean copulas with one parameter. This method naturally encompasses the special case in which all the bivariate copulas present the same parameter. The objective is to generate variates from the distribution function

$$F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ...F_n(x_n))$$

where C is a copula function. If C is Archimedean, it admits this representation:

$$C(u_1, u_2, ..., u_n) = \phi^{-1}(\phi(u_1) + ... + \phi(u_n)),$$

where ϕ is called the *generator* of the Archimedean copula. Genest (1987), Genest and Rivest (1993) and Lee (1993) showed how to generate full distributions by recursively simulating conditional ones. Assume a joint probability density function of a multivariate distribution $(X_1, X_2, ..., X_n)$ exists. Then, for $i = 2, ..., n$ the density function of $X_1, ..., X_i$ can be written as:

$$f_i(x_1, x_2, ..., x_i) = \frac{\partial^i}{\partial x_1 ... \partial x_i} \phi^{-1}\{(\phi(F_1(x_1)) + ... + \phi(F_i(x_i))\} =$$

$$= \phi^{-1(i)}\{(\phi(F_1(x_1)) + ... + \phi(F_i(x_i))\} \prod_{j=1}^{i} \phi^1\left[F_j(x_j)\right] F_j^{(1)}(x_j)$$

where the superscript (j) denotes the j-th partial derivative. Hence, we can compute the conditional density of X_i given $X_1, ..., X_{i-1}$

$$F_i(x_i|x_1, ..., x_{i-1}) = \frac{f_i(x_1, ..., x_i)}{f_{i-1}(x_1, ..., x_{i-1})} =$$

$$= \phi^{(1)}[F_i(x_i)]F^{(1)}(x_i)\frac{\phi^{-1(i-1)}\{(\phi(F_1(x_1)) + ... + \phi(F_i(x_i))\}}{\phi^{-1(i-1)}\{(\phi(F_1(x_1)) + ... + \phi(F_{i-1}(x_{i-1}))\}}$$

Then, we obtain the conditional distribution function of X_i given $X_1, ..., X_{i-1}$:

$$F_i(x_i|x_1, ..., x_{i-1}) = \int_{-\infty}^{x_i} f_i(x|x_1, ..., x_{i-1})dx =$$

$$= \frac{\phi^{-1(i-1)}\{(c_{i-1} + \phi(F_i(x_i))\}}{\phi^{-1(i-1)}(c_{i-1}}$$

where $c_i = \phi[F_1(x_1)] + ... + \phi[F_i(x_i)]$.

Starting from these considerations, we present the algorithm that Lee (1993)proposed to generate outcomes from an n-dimensional Archimedean copula:

1. Generate n independent uniform random numbers $U_i \sim U[(0, 1)]$ for $i = 1, ...n$
2. Set $X_1 = F_1^{-1}(U_1), c_0 = 0$.
3. Calculate X_i for $i = 2, ...n$ recursively exploiting the properties of the conditional distribution:

$$U_i = F_i(X_i|x_1, ..., x_{i-1}) = \frac{\phi^{-1(i-1)}\{(c_{i-1} + \phi(F_i(x_i))\}}{\phi^{-1(i-1)}(c_{i-1})}$$

7.2 Generating 4-dimensional Archimedean copulas

In the following subsections we report the algorithms to generate the 4-dimensional copulas we used in the paper: the Clayton, the Frank and the Gumbel ones. Please notice that closed form solutions can be obtained for the former, while for the Frank and the Gumbel copula, numerical methods are necessary to recover the variates. Throughout the section, θ_1, θ_2 and θ_3 refer to the corresponding bivariate copula parameter between dimension 1 and, respectively, dimensions 2, 3 and 4, while $(U_1, ...U_4)$ refers to a 4-dimensional vector where where each of the U_i's, $i = 1, .., 4$ generated independently from a uniform distribution with values in (0,1). For the copulas we considered, these parameters can be obtained simply from their relationship with the sample Kendall's τ:

1. Clayton: $\theta = \frac{2\tau}{1-\tau}$
2. Gumbel: $\theta = \frac{1}{1-\tau}$
3. Frank: $\tau = 1 - \frac{4}{\theta}[D_1(-\theta) - 1]$, where D denotes the Debye function of order 1.

Notice that the parameters for Frank's copula are obtained numerically finding the zero of the equation that links them to Kendall's τ.

7.2.1 4 dimensional Clayton copula

First, we recall the generator of the Clayton copula:

$$\phi(t) = t^{-\theta} - 1$$

We then compute its inverse and its first three derivatives:

$$\phi^{-1}(s) = (1+s)^{-\frac{1}{\theta}}$$

$$\phi^{-1(1)}(s) = -\frac{1}{\theta}(1+s)^{-\frac{1}{\theta}-1}$$

$$\phi^{-1(2)}(s) = -\frac{1}{\theta}(-\frac{1}{\theta} - 1)(1+s)^{-\frac{1}{\theta}-2}$$

$$\phi^{-1(3)}(s) = -\frac{1}{\theta}(-\frac{1}{\theta} - 1)(-\frac{1}{\theta} - 2)(1+s)^{-\frac{1}{\theta}-3}$$

Then, after having drawn $(U_1, ..., U_4)$, we compute the random variates from the Clayton copula in the following way:

$$X_1 = F_1^{-1}(U_1)$$

$$X_2 = F_2^{-1}\left[\sqrt[\theta_1]{\frac{1}{\left[\sqrt[\frac{1}{\theta_1}+1]{\frac{1}{U_2}} - 1\right] F_1(x_1)^{-\theta_1} + 1}}\right]$$

$$X_3 = F_2^{-1}\left[\frac{1}{\{\left[\sqrt[\frac{1}{\theta_2}+2]{\frac{1}{U_3}} - 1\right][F_1(x_1)^{-\theta_2} + F_2(x_2)^{-\theta_2} - 1] + 1\}^{\frac{1}{\theta_2}}}\right]$$

$$X_4 = F_4^{-1}\left[\sqrt[\theta_3]{\frac{1}{\left[\sqrt[\frac{1}{\theta_3}+3]{\frac{1}{U_4}}-1\right]\left[F_1(x_1)^{-\theta_2}+F_2(x_2)^{-\theta_2}+F_3(x_3)^{-\theta_3}-2\right]+1}}\right]$$

As we remarked above, the form of the Clayton copula permits us to find these analytical expressions for (X_1, X_2, X_3, X_4) generated with the Genest and Rivers' approach.

7.2.2 4-dimensional Frank copula

Frank's copula generation is somewhat more difficult than the generation of the Clayton copula. We recall the copula generator, its inverse and its derivatives up to the third order:

$$\begin{aligned}
\phi(t) &= -\ln\frac{e^{-\alpha t}-1}{e^{-\alpha}-1} \\
\phi^{-1}(s) &= \frac{1}{\alpha}\ln\left[1+e^{s}(e^{\alpha}-1)\right] \\
\phi^{-1(1)}(s) &= -\frac{1}{\alpha}\frac{1}{1+e^{s}(e^{\alpha}-1)}(e^{\alpha}-1)e^{s} \\
\phi^{-1(2)}(s) &= \frac{1}{\alpha}\frac{1}{(1+e^{s}(e^{\alpha}-1))^2}(e^{\alpha}-1)e^{s} \\
\phi^{-1(3)}(s) &= \frac{1}{\alpha}\frac{1-e^{s}(e^{\alpha}-1)}{(1+e^{s}(e^{\alpha}-1))^3}(e^{\alpha}-1)e^{s}
\end{aligned}$$

From the expressions we derive below, it is easy to see that the high non-linearity of the equations does not allow us to obtain closed-form solutions, contrary to the Clayton copula:

$$\begin{aligned}
X_1 &= F_1^{-1}(U_1) \\
U_2 &= \frac{e^{\theta_1 F_1(x_1)}\left[e^{\theta_1 F_1(x_1)}-1\right]}{e^{\theta_1}-1+\left[e^{\theta_1 F_1(x_1)}-1\right]\left[e^{\theta_1 F_2(x_2)}-1\right]} \\
U_3 &= \frac{\left[e^{\theta_2 F_3(x_3)}-1\right]\left[e^{\theta_2}-1\right]\left\{e^{\theta_2}-1+\left[e^{\theta_2 F_1(x_1)}-1\right]\left[e^{\theta_2 F_2(x_2)}-1\right]\right\}^2}{\left\{\left[e^{\theta_2 -1}\right]^2+\left[e^{\theta_2 F_1(x_1)}-1\right]\left[e^{\theta_2 F_2(x_2)}-1\right]\left[e^{\theta_2 F_3(x_3)}-1\right]\right\}^2} \\
U_4 &= \frac{\left[e^{\theta_3 F_4(x_4)}-1\right]\left[e^{\theta_3}-1\right]\left\{\left[e^{\theta_3}-1\right]^3-P\right\}\left\{\left[e^{\theta_3}-1\right]^2+Q\right\}^3}{\left\{\left[e^{\theta_3}-1\right]^3+P\right\}^3\left\{\left[e^{\theta_3}-1\right]^2-Q\right\}}
\end{aligned}$$

where

$$P = \prod_{i=1}^{4} e^{\theta_3 F_i(x_i)} - 1$$

and

$$Q = \prod_{i=1}^{3} e^{\theta_3 F_i(x_i)} - 1$$

We find the zeros of the above equations to obtain (X_1, X_2, X_3, X_4).

7.2.3 4 dimensional Gumbel copula

As for the Frank copula, the high non-linearity of the relationship between U_i's and the X_i's do not permit to obtain analytical expressions. Resorting to numerical methods, anyway, one can easily solve the equations derived from the properties of conditional distributions and generate variates from a 4 dimensional Gumbel copula. First, we recall the generator of the bivariate Gumbel copula with parameter θ, its inverse and the derivative of the latter up to the third order:

$$\phi(t) = (-\ln t)^{\theta}$$

$$\phi^{-1}(s) = e^{-t^{\frac{1}{\alpha}}}$$

$$\phi^{-1(1)}(s) = -\frac{1}{\alpha} t^{\frac{1}{\alpha}-1} e^{-t^{\frac{1}{\alpha}}}$$

$$\phi^{-1(2)}(s) = -\frac{1}{\alpha^2} t^{\frac{1}{\alpha}-2} e^{-t^{\frac{1}{\alpha}}} (1 - \alpha - t^{\frac{1}{\alpha}})$$

$$\phi^{-1(3)}(s) = -\frac{1}{\alpha^3} t^{\frac{1}{\alpha}-3} e^{-t^{\frac{1}{\alpha}}} (1 - 3\alpha + (2\alpha - 3) t^{\frac{1}{\alpha}} + 2t^{\frac{2}{\alpha}})$$

Applying Genest and River' s procedure we get:

$$X_1 = F_1^{-1}(U_1)$$

$$U_2 = \frac{e^{-[\phi(F_1(x_1)) + \phi(F_2(x_2))]^{\frac{1}{\theta_1}}} [\phi(F_1(x_1)) + \phi(F_2(x_2))]^{\frac{1}{\theta_1}-1}}{e^{-\phi(F_1(x_1))^{\frac{1}{\theta_1}}} \phi(F_1(x_1))^{\frac{1}{\theta_1}-1}}$$

$$U_3 = \frac{e^{-[P+\phi(F_3(x_3)]^{\frac{1}{\theta_2}}} [P + \phi(F_3(x_3))]^{\frac{1}{\theta_2}-2} \left[1 - \theta_2 - (P + \phi(F_3(x_3)))^{\frac{1}{\theta_2}}\right]}{e^{-P^{\frac{1}{\theta_2}}} P^{\frac{1}{\theta_2}-2} \left[1 - \theta_2 - P^{\frac{1}{\theta_2}}\right]}$$

$$U_4 = \frac{e^{-[Q+\phi(F_4(x_4))]^{\frac{1}{\theta_3}}} [Q + \phi(F_4(x_4))]^{\frac{1}{\theta_3}-3} \left[2\theta_3^2 - 3\theta_3 + 1 + (2\theta_3 - 3)(Q + \phi(F_4(x_4)))^{\frac{1}{\theta_3}} + 2(Q + \phi(F_4(x_4)))^{\frac{2}{\theta_3}}\right]}{e^{-Q^{\frac{1}{\theta_3}}} Q^{\frac{1}{\theta_3}-3} \left[2\theta_3^2 - 3\theta_3 + 1 + (2\theta_3 - 3) Q^{\frac{1}{\theta_3}} + 2Q^{\frac{2}{\theta_3}}\right]}$$

where

$$P = \sum_{i=1}^{2} \phi(F_i(x_i))$$

and

$$Q = \prod_{i=1}^{3} \phi(F_i(x_i))$$

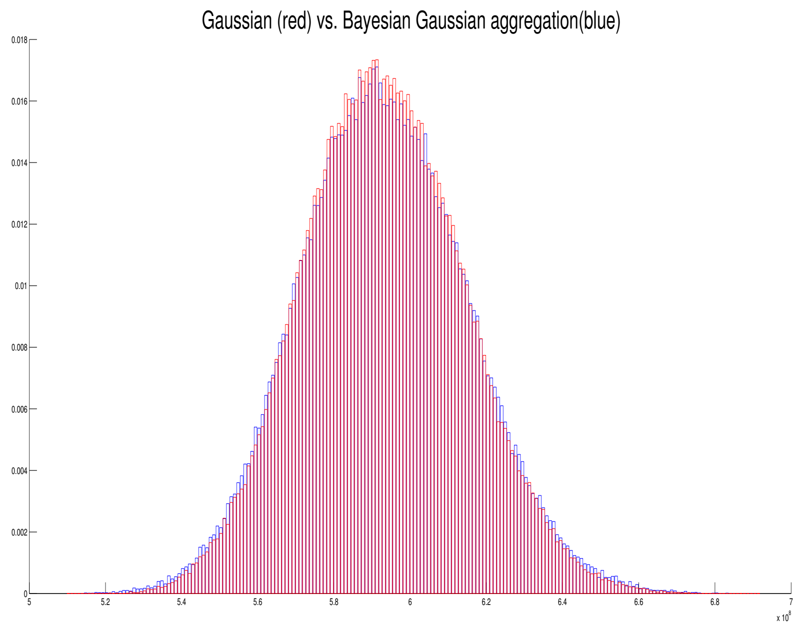

Figure 1: This Figure shows the density of the predictive distribution of OLLs obtained using a Gaussian copula with correlation matrix estimated from company data on losses (red) and using the Bayesian Gaussian model (blue).

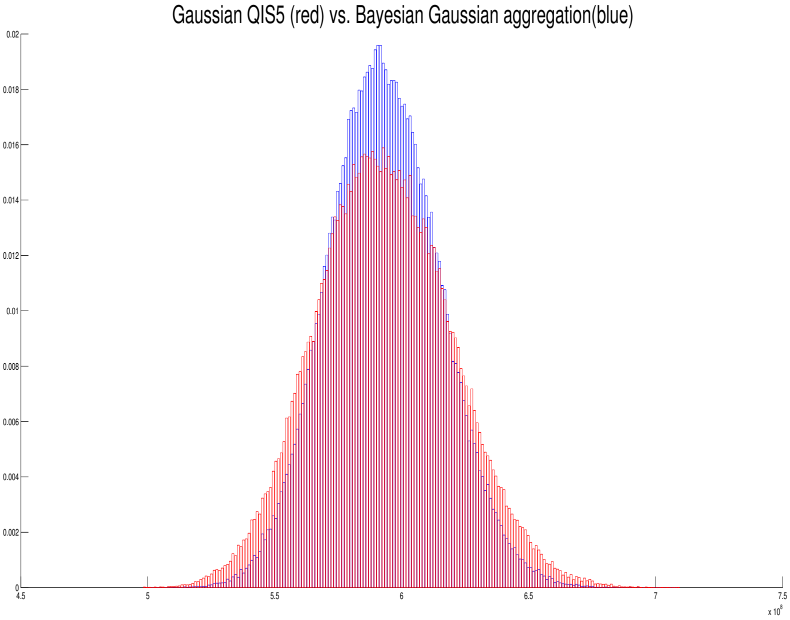

Figure 2: This Figure shows the density of the predictive distribution of OLLs obtained using a Gaussian copula with the correlation matrix given by the CEIOPS (red) and using a Bayesian Gaussian model (blue).